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Electron tunnelling through a self-similar fractal potential on the generalized Cantor set

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Abstract. A proper formalism developed earlier to study electron tunnelling through a self-similar fractal potential (SSFP) posed on the Cantor set is extended here to describe the SSFP whose levels consist of N fractals of the next level. We have derived a functional equation for the transfer matrix of this potential and found three different solutions. Two of them correspond to SSFP barriers and SSFP wells whose power may be arbitrary. The third one relates to the only SSFP barrier whose power has a definite value. These solutions show that SSFPs, in the general case, are approximately scale invariant in the long- and short-wave regions, and only the limiting SSFP whose fractal dimension is equal to unity should be strictly scale invariant. We have shown that except for the limiting case the tunnelling parameters of SSFPs, with the same fractal dimension depend on N. In addition, we have established a link between the solutions of the functional equation and the power of SSFPs.

1. Introduction

The general aim of this paper is to develop and generalize the method described in [1] for studying electron tunnelling through a self-similar fractal potential (SSFP) given on the generalized Cantor set. Recall that the SSFP, like the Cantor set itself where it defined, represents a hierarchical structure consisting of an infinite number of levels. In [1] the case where each level of the fractal consists of two fractals of the next level was considered. For this structure, on the basis of the transfer matrix method (TMM) [2], recurrence relations for the transfer matrices (TMs) of neighbouring levels were obtained (see also [3, 4]).

It is known (see, for example, [2]) that in the case of Euclidean geometry, the recurrence relations for TMs enable us to make a comprehensive description of the tunnelling parameters for many-barrier structures. However, for the SSFP we have another situation. This structure is devoid of a smallest structure element. Therefore, the traditional use of such relations becomes impossible when one attempts to calculate its tunnelling parameters. To do this, additional relations for the TMs are needed.

It is evident that missing relations may be derived when the problem investigated possesses some symmetry. So, for example, explicit expressions (see [5]) for the TM of a system of N identical barriers were obtained when recurrence relations for the matrices were used together with the translational symmetry. At first glance, for the SSFP such a role should be played by scale invariance. However, as was shown in [1], strictly speaking, the SSFP does not possess, this symmetry: there is no scale transformation to be valid simultaneously for the TMs of all of its levels. At the same time one may speak about the partial scale invariance of the SSFP because the TMs of its first two levels, as functions of the electron wavenumber, are connected by a scale transformation. The latter not only satisfies the natural physical requirements stated in [1], but it also transforms into the symmetry condition for the limiting case when the fractal dimension of the SSFP is equal to unity. As was shown in [1], the limiting SSFP is scale invariant. The corresponding scaling relation describes all of its levels without exception. In this case, it may be obtained immediately from the Lippmann–Schwinger equation which is an integral analogue of the Schrödinger equation.

So, in the general case the TMs of the zero and first levels must obey two equations. As was shown in [1], this system is reduced to a functional equation for the zero-level TM, that is, for the TM of the initial SSFP. This equation proved to have three classes of solutions: two of them have one parameter (they describe TMs which are non-analytic in the long-wave region and are characterized here by the fractal dimension); the third solution, which is free of parameters, describes an analytical TM. It is remarkable that in the above-mentioned limit a solution for this equation is the TM of the δ-potential. Taking into account the scale invariance of the limiting SSFP, we may conclude that all levels of the limiting SSFP scatter an electron with a definite energy, like δ-potentials (the power of the δ-potentials (and SSFPs) is reduced to half with increasing level number).

In this paper we consider a more complicated model when the SSFP is taken on a generalized Cantor set, every level of which consists of N ($N \geqslant 2$) SSFPs of the following level. Such a model enables us to study the dependence of the physical properties of the fractal on its lacunarity, in particular on the number of lacunas, in our case (here the term 'lacuna' means a gap between Cantor segments; in our model each level of the Cantor set consists of N Cantor segments and $N - 1$ lacunas). At present this question has not yet been satisfactorily investigated. Although there are some works considering the physical (see, for example, [6–8]) and mathematical (see [9] and references therein) aspects of this problem. It is of great interest, in this case, to compare the physical properties of fractals having the same fractal dimension but a different number of lacunas. In addition, for this model it is important to examine the properties of the limiting SSFP when the fractal dimension of the corresponding Cantor set is equal to unity.

In this paper we also display a link between the solutions of the functional equations for TMs and the parameters of the SSFP such as its power and width. In [1] this question remained to be solved.

2. The recurrence relations for the tunnelling parameters

We must recognize that although the SSFP has an unusual geometry, stationary electron states for this structure must be described by the probability-flow density being constant everywhere along the OX-axis. In particular, this quantity must have the same value in all the out-of-barrier regions (OBRs) of the SSFP (recall that, by definition (see [1, 2]), OBRs are regions where a potential is equal to zero; in our problem OBRs coincide with lacunas). The TM formalism of the one-dimensional Schrödinger equation (OSE) guarantees entirely the fulfilment of this requirement. Therefore, as in [1], we will treat this problem on the basis of the OSE. The only distinction from [1] lies in the fact that we will now consider the case when each SSFP of the nth level represents a symmetrical system of N, rather than two, identical barriers (SSFPs of the $(n+1)$th level). So, if L is the width of the whole structure, and α is the scale factor ($\alpha > N$), then the fractal width d_n of the nth level is equal to L/α^n, and the width l_n of the corresponding OBRs is γd_{n-1}, where $\gamma = \frac{\alpha - N}{\alpha(N-1)}$, $n = 1, 2, \ldots$.

Note that the recurrence relations derived in [1] are unfit for this case. It is more suitable here to use the expressions obtained in [5]. As is shown there, the (real) transmission coefficient $T_{\{1,N\}}$ and phases $J_{\{1,N\}}$ and $F_{\{1,N\}}$ of a system of N identical barriers are described by expressions (the notation is taken from [5]; see also [2])

$$T_{\{1,N\}} = \left(1 + \rho\omega_N^2\right)^{-1} \qquad J_{\{1,N\}} = \cos^{-1}\left(\sqrt{T_{\{1,N\}}}\theta_N\right) - kl \tag{1}$$

$$F_{\{1,N\}} = \begin{cases} \tilde{F}_1 & \text{if} \quad \omega_N \geqslant 0 \\ \tilde{F}_1 + \pi & \text{if} \quad \omega_N < 0 \end{cases}$$

where l is the distance between the nearest barriers; $\rho = \tilde{R}_1/\tilde{T}_1$;

$$\omega_N = \frac{\sinh(N\eta)}{\sinh(\eta)} \qquad \theta_N = [\mathrm{sgn}(u)]^N \cosh(N\eta)$$

$$\eta = \cosh^{-1}|u| \qquad \text{if} \quad |u| > 1$$

or

$$\omega_N = \frac{\sin(N\eta)}{\sin(\eta)} \qquad \theta_N = \cos(N\eta)$$

$$\eta = \cos^{-1}(u) \qquad \text{if} \quad |u| \leqslant 1 \qquad u = \frac{\cos(\tilde{J}_1 + kl)}{\sqrt{\tilde{T}_1}}.$$

Here $\tilde{T}_1$, $\tilde{J}_1$ and $\tilde{F}_1$ are the tunnelling parameters of one barrier.

In accordance with (1), recurrence relations for the $(n-1)$th level of the SSFP may be written in the form

$$\frac{\cos(J_n + kl_n)}{\sqrt{T_n}} = \frac{1}{2}(\beta_n + \beta_n^{-1}) \tag{2}$$

$$\frac{\cos(J_{n-1} + kl_n)}{\sqrt{T_{n-1}}} = \frac{1}{2}(\beta_n^N + \beta_n^{-N}) \tag{3}$$

$$\frac{R_{n-1}}{T_{n-1}} = \frac{R_n}{T_n}\Omega_n^2 \qquad \Omega_n = \frac{\beta_n^N - \beta_n^{-N}}{\beta_n - \beta_n^{-1}} \tag{4}$$

$$F_{n-1} = \begin{cases} F_n & \text{if} \quad \Omega_n \geqslant 0 \\ F_n + \pi & \text{if} \quad \Omega_n < 0 \end{cases} \tag{5}$$

where T_n, J_n and F_n are the tunnelling parameters (the transmission coefficient and phases) of an SSFP of the nth level; $R_n = 1 - T_n$, β is an auxiliary quantity to be defined from equation (2). All of these index quantities are functions of the wavenumber k. Our aim is to find them.

As in the particular model [1], an important role in our formalism is played by the limiting case when $\alpha = N$. At first glance, it is unrelated to fractals since the well known procedure used for constructing the SSFP does not alter (at exact equality $\alpha = N$) the initial rectangular barrier (well). However, there is another case there. We may consider the limit of the sequence of SSFPs $V(x;\alpha)$ at $\alpha \to N+0$. It is obvious that the limiting structure as well as each member of this sequence should be a fractal one. In particular, the corresponding Cantor set should be, as before, of the zero measure. At the same time its fractal dimension is equal to unity! It is explained by the fact that despite the length l_n (for fixed n) tending to zero in this case, the sum of all the removed gaps, as well as for each member of this sequence, remains equal to L.

The existence of this limit is demonstrated by the fact that recurrence relations (2)–(5) have as a solution not only the tunnelling parameters of the rectangular barrier, but also those of the δ-potential (see also [1])

$$T_n(k) = (1 + p_n^2)^{-1} \qquad J_n(k) = -\tan^{-1}(p_n)$$
$$F_n = \begin{cases} 0 & \text{if} \quad W \geqslant 0 \\ \pi & \text{if} \quad W < 0 \end{cases} \qquad p_n = \frac{mW}{\hbar^2 k N^n} \tag{6}$$

where m is the electron mass, $\hbar$ is Planck's constant, and W is a power.

As can be seen from expression (6), these functions also obey the scaling relations

$$T_n(k) = T_{n-1}(Nk) \qquad J_n(k) = J_{n-1}(Nk) \qquad F_n(k) = F_{n-1}(Nk). \tag{7}$$

The same relations can also be obtained for the limiting SSFP (as in the particular case [1], it is better to use the Lippmann–Schwinger equation for this purpose, taking into account that the neighbouring levels are connected by relations $V_n(x) = V_{n-1}(Nx)$). These facts show that: (a) the limiting SSFP exists; (b) its tunnelling parameters coincide with those of the δ-potentials; (c) the limiting SSFP is scale invariant.

In the general case, to solve recurrence relations (2)–(5) is more complicated problem. At $\alpha \neq N$, the SSFP is no longer scale-invariant, and its tunnelling parameters do not coincide with those of the δ-potential.

As in [1], we will suppose that for the SSFP in the general case relations the following:

$$T_1(k) = T_0(\alpha k) \qquad y_1(k) = y_0(\alpha k) \qquad F_1(k) = F_0(\alpha k) \tag{8}$$

where $y_n = \frac{1}{2}\pi - J_n$ for all n. This condition does not destroy the link, between the first two levels, determined by relations (2)–(5), and, in addition, it coincides at $\alpha = N$ with (7) for the first two levels.

To find TMs obeying both recurrence relations (2)–(5) and condition (8), we have to derive relations to be inverse to (2)–(5). It is easy to check that the required relations may be written in the form

$$T_n = \mathcal{L}_1(kd_{n-1}, T_{n-1}, y_{n-1}) \equiv \frac{T_{n-1}}{T_{n-1} + R_{n-1}\tilde{\Omega}_{n-1}^2} \tag{9}$$

$$y_n = \mathcal{L}_2(kd_{n-1}, T_n, y_{n-1}) \equiv \gamma k d_{n-1} + \operatorname{sgn}(\tan(B_{n-1}))\sin^{-1}\left(\sqrt{T_n}\theta_{n-1}\right) \tag{10}$$

where

(a) $$\tilde{\Omega}_{n-1} = \frac{\sinh(\Phi_{n-1}/N)}{\sinh(\Phi_{n-1})} \qquad \theta_{n-1} = \cosh\left(\frac{\Phi_{n-1}}{N}\right)$$

$$\Phi_{n-1} = \cosh^{-1}|u_{n-1}| \qquad \text{if} \qquad |u_{n-1}| > 1$$

or

(b) $$\tilde{\Omega}_{n-1} = \frac{\sin(\Phi_{n-1}/N)}{\sin(\Phi_{n-1})} \qquad \theta_{n-1} = \cos\left(\frac{\Phi_{n-1}}{N}\right)$$

$$\Phi_{n-1} = \cos^{-1}|u_{n-1}| \qquad \left(0 \leqslant \Phi_{n-1} \leqslant \tfrac{1}{2}\pi\right) \qquad \text{if} \quad |u_{n-1}| \leqslant 1$$

$$u_{n-1} = \frac{\sin(B_{n-1})}{\sqrt{T_{n-1}}} \qquad B_{n-1} = y_{n-1} - \gamma k d_{n-1}.$$

As for the limiting SSFP, we will assume that

$$F_n = \begin{cases} 0 & \text{if} \quad W \geqslant 0 \\ \pi & \text{if} \quad W < 0 \end{cases}$$

where W is the power of the SSFP ($W = \int_0^L V(x)\,\mathrm{d}x$). In deriving these relations we have taken into account that the phases J and F, in the TM formalism, are sufficient to be defined in the interval $[0, \pi]$. For the simultaneous variations in values of both these quantities by π influences only the sign of a TM, and, as a consequence, the sign of a wavefunction in the corresponding OBRs. At the same time, physically valuable quantities, such as the probability density and probability-flow density, remain unchanged in this case.

3. The functional equations for the tunnelling parameters of the SSFP

Considering (8), we may now write down relations (9) and (10) at $n = 1$ in the form

$$T_0(\alpha k) = \mathcal{L}_1(kL, T_0(k), y_0(k)) \tag{11}$$

$$y_0(\alpha k) = \mathcal{L}_2(kL, T_0(\alpha k), y_0(k)). \tag{12}$$

Or, introducing the dimensionless variable ϕ ($\phi = kL$), we have

$$T(\alpha\phi) = \mathcal{L}_1(\phi, T(\phi), y(\phi)) \tag{13}$$

$$y(\alpha\phi) = \mathcal{L}_2(\phi, T(\alpha\phi), y(\phi)) \tag{14}$$

(the index '0' is omitted here).

Like the particular case [1], functional equations (13) and (14) possess three families of solutions; two of them depend on a parameter and the third solution is free of parameters. The one-parameter solutions have the fractal asymptotes in the long-wave region

$$T_0(\phi) = b^2\phi^{2s} \qquad y_0(\phi) = b\phi^s \tag{15}$$

where $s = \ln(N)/\ln(\alpha)$; these families differ from each other by the sign of the parameter b. The third solution describes the TM which is analytic at the point $\phi = 0$,

$$T_0(\phi) = a^2\phi^2 \qquad y_0(\phi) = b\phi \qquad (b < 0) \tag{16}$$

where the parameters a and b obey the equations

$$\alpha = \frac{\sinh(\Phi)}{\sinh(\Phi/N)} \qquad b\alpha - \gamma = -\alpha a\cosh(\Phi/N) \qquad b - \gamma = -a\cosh(\Phi). \tag{17}$$

To solve analytically (with respect to Φ) the first equation in (17) is impossible in the general case. However, it is seen graphically that at $\alpha > N$ it has the only solution. Consequently, it provides a unique solution of equations (13) and (14), with asymptotes (16). Note that these equations, for $b > 0$, have another asymptote of type (16). However, it has proved to be unstable, as in [1], with respect to the iterative procedure of solving the functional equations.

Let us introduce the auxiliary functions $t_n(\phi)$ and $f_n(\phi)$:

$$t_n(\phi) = \mathcal{L}_1(\phi/\alpha, t_{n-1}(\phi/\alpha), f_{n-1}(\phi/\alpha)) \tag{18}$$

$$f_n(\phi) = \mathcal{L}_2(\phi/\alpha, t_n(\phi), f_{n-1}(\phi/\alpha)) \tag{19}$$

$n = 0, 1, \ldots$. Then the solutions of equations (13) and (14) may be written in the form (see [1, 10])

$$T(\phi) = \lim_{n\to\infty} t_n(\phi) \qquad y(\phi) = \lim_{n\to\infty} f_n(\phi). \tag{20}$$

Taking all the above-mentioned asymptotes as the initial functions $t_0(\phi)$ and $f_0(\phi)$, we may obtain, for any value of ϕ, all three solutions of functional equations (13) and (14).

As follows from the theory of functional equations (see [10]), we might take as $t_0(\phi)$ and $f_0(\phi)$ other functions satisfying conditions $t_0(0) = 0$, $f_0(0) = 0$. In any case, the iterative procedure (18)–(20) yields solutions with one of asymptotes (15) or (16). The absence of explicit expressions for the parameters a and b in (16) does not represent any obstacle in calculating the functions with these asymptotes. We might take any real values of a and b, and then use the iterative procedure (18)–(20) to provide the same two functions with the asymptotes (16).

4. The connection between the solutions of the functional equations and the SSFP characteristics

Now we have to connect the parameters a and b of the obtained TMs to the power and width of the corresponding SSFPs. To solve this task, let us address the procedure of constructing the SSFP (see, for example, [4, 6]). As can be seen, the SSFP power, at any values of the parameters α and N, is equal to the area of the rectangular barrier (or well) from which the SSFP is constructed. So, the infinite set of SSFPs constructed from this rectangular barrier (well), at different values of these parameters, should be described by the same power.

This is also true for the limit $\alpha \to N$. As was to be expected, the tunnelling parameters of the δ-potential are solutions to the functional equations, in this case.

At small values of k we have for this potential that

$$\sqrt{T} \approx -y \approx \frac{\hbar^2 k}{mW}. \tag{21}$$

Moreover, one can check that for the SSFP, in the limit $\alpha \to N$, we have $s = 1$ in (15), and $b = -a = -(3N)^{-1}$ in (16). In this limit all of these asymptotes would be coincide with those (21) for the δ-potential. As a result, for the solutions with fractal asymptotes, we have

$$b = -\frac{\hbar^2}{mWL} \tag{22}$$

where W and L may have any values. As can be seen, this asymptote does not depend on N. At the same time the SSFP characterized by the 'Euclidean' asymptotes in the long-wave region should have a quite definite power at the given fractal width:

$$W = \frac{3N\hbar^2}{mL} > 0 \tag{23}$$

and it depends on the parameter N.

5. Main results and conclusions

We have shown that the correct approach [1] developed earlier to study electron tunnelling for the particular case of the SSFP, with each level consisting of two SSFPs of the next level

($N = 2$), can be generalized to the case of an arbitrary value of N. We have found that, as in the particular case, there are three classes of SSFPs whose TMs obey condition (8). Two of them are formed from SSFP barriers and SSFP wells with any power. The third class is presented, for a given Cantor set, by the only SSFP barrier whose power is determined by the expression (23). The properties of this potential differ qualitatively, in the long-wave region, from those of the SSFPs of the first two classes.

The TMs for the first two classes are non-analytic at the point $\phi = 0$. The behaviour of their tunnelling parameters in the long-wave region is characterized by the fractal dimension. The role of OBRs (or lacunas) in this case has proved to be inessential. At the same time, the TM of the third class is analytic at this point. The phase path of the electron wave in some OBR, which is equal to $\gamma\phi$, is comparable with that in the corresponding barrier region, that is, with $y(\phi)$. Thus, we have every reason to believe that the inherent feature of this SSFP is that the barrier regions in this case are, figuratively speaking, as 'empty' as the OBRs, that is, the SSFP of the third class differs from zero only on a countable subset of the Cantor set, rather than on the whole one (as takes place for the SSFPs of the first two classes). Hence, we may assume that this SSFP consists, in fact, of the δ-potentials posed at the endpoints of the Cantor set. Of course, this question needs in a more detailed treatment.

Let us note that the SSFPs of the first two classes may have different values of N and α but the same fractal dimension, and, consequently, such potentials must have the same long- and short-wave asymptotes. Of course, over the whole ϕ-axis the tunnelling parameters for them are different. To illustrate some interesting peculiarities of SSFPs, we have performed numerical calculations of the function $T(\phi)$ at several values of the parameters N, α and b. As expected, structures with the same fractal dimension, but with a different number of lacunas, have, in the general case, a different transparency (see figures 1–3). This means that the fractal dimension is a rough enough fractal characteristic. As can be seen, the larger N, the smoother is the function $T(\phi)$. It is explained by the fact that, for any value of ϕ, the distance between the points ϕ and $\alpha\phi$, which are connected by (11), increases together with N.

It is not difficult to show analytically that the SSFP becomes more transparent at large values of the electron energy. So, assuming in (13) $R \ll 1$, we may obtain for the envelope $\tilde{R}(\phi)$ of the function $R(\phi)$ that $\tilde{R}(\alpha\phi) \approx N^{-2}\tilde{R}(\phi)$ (note that $R(\phi)$ takes maximal values at the points where the parameter $|u|$ is equal to unity). It follows from here that $\tilde{R}(\phi) \sim \phi^{-2s}$ at large values of ϕ. This asymptote is true for the third-class SSFP as well.

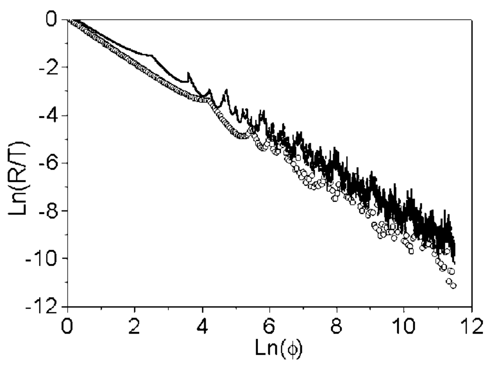

Figure 1. The function $\ln(R/T)$ as a function of ϕ for the first solution at $\beta = 1$, $N = 2$, $\alpha = 3$ for the full curve and $N = 4$, $\alpha = 9$ for the curve of open circles.

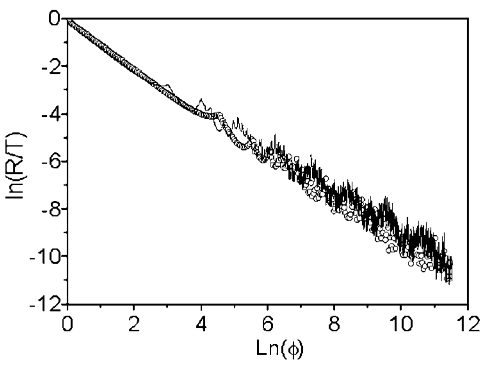

Figure 2. The same as for figure 1 but for the second solution; $\beta = -1$.

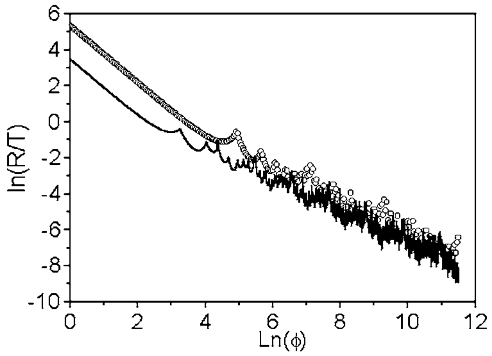

Figure 3. The same as for figure 1 but for the third solution.

We also point out the fact (see also [1]) that the scale invariance in the case of SSFP manifests itself approximately. Strictly speaking, this potential does not possess this symmetry since a link between levels, expressed by the recurrence relations, cannot be reduced to the same functional equation, for all levels of the SSFP. As was pointed out in [1], it is explained, in the last analysis, by the fact that the SSFP is characterized by two length scales, rather than one (unlike the Cantor set itself). The second scale is associated with the potential power; the smaller the power of the SSFP, the larger this scale. Thus, the scale invariance may appear in those regions on the ϕ-axis where one of the scales dominates. We have shown that, in the general case, it takes place in the long- and short-wave regions. The limiting SSFP is strictly scale invariant because it is described only by one scale (see also [1]).

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